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2010 J. Phys. A: Math. Theor. 43 175207

(http://iopscience.iop.org/1751-8121/43/17/175207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 175207 (9pp) doi:10.1088/1751-8113/43/17/175207

Second-degree discrete Painlevé equations conceal first-degree ones

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Received 27 January 2010, in final form 1 February 2010
Published 14 April 2010
Online at stacks.iop.org/JPhysA/43/175207

Abstract

We examine various second-degree difference equations which have been proposed over the years and according to their authors' claims should be integrable. This study is motivated by the fact that we consider that second-degree discrete systems cannot be integrable due to the proliferation of the images (and pre-images) of the initial point. We show that in the present cases no contradiction exists. In all cases examined, we show that there exists an underlying integrable first-degree mapping which allows us to obtain an appropriate solution of the second-degree one.

PACS numbers: 02.30.Ik, 04.60.Nc, 05.45.−a

1. Introduction

The name of Painlevé equations usually evokes one of the six equations discovered by Painlevé and his school. The importance of these equations lies in the fact that their solutions introduce new functions, the Painlevé transcendents. Painlevé [1] obtained the equations bearing his name while classifying second-order differential equations of the form

$$w'' = f(w', w, t) \tag{1.1}$$

from the point of view of integrability, based on the criterion which in modern parlance would be qualified as singularity analysis. The results of Painlevé spurred further studies aiming at the discovery of new transcendents. The works of Chazy [2], Garnier [3], more recently those of Bureau [4] and even more recently the works of Cosgrove [5] spring to mind. These studies did not lead to the discovery of new transcendents despite the fact that they studied equations beyond the class of those analysed by Painlevé and his school. In particular, Chazy examined equations of second degree in the second derivative. The general form of these equations is

$$(w'')^2 = f(w', w, t)w'' + g(w', w, t). \tag{1.2}$$

Chazy's study was only recently completed by Cosgrove, but still the complete classification of equations of the form (1.2) possessing the Painlevé property is not available. The existence of a quadratic term in (1.2) and the fact that one must consider square roots in order to integrate the equation does not pose a serious problem. While integrating (1.2), one can follow a solution by analytic continuation and the only points that would cause difficulties are the points where a singularity that induces multivaluedness appears. But such singularities are absent by definition in a system which possesses the Painlevé property.

The situation is different where discrete Painlevé equations are concerned. The analogues of the Chazy–Cosgrove equations do exist; however, their second-degree form is problematic. We suggest that those cases which are integrable should be expressible as first-degree equations. Second-degree equations lead generally to multivaluedness introduced by square roots: the iteration of initial data generically leads to an exponentially increasing number of branches. In [6] we have studied various correspondences and concluded that the proliferation of images (and preimages) of a given point is incompatible with integrability. In the light of this remark, how can one interpret the recent results of [7], where second-degree, second-order mappings were derived and where it was argued that they were integrable? It is the aim of this paper to show that these systems are integrable indeed and that with the adequate parametrization they can be expressed as first-degree ones.

2. Some examples of hidden first-degree systems

Before proceeding to examples of second-degree mappings, we clarify our terminology. In what follows, we shall limit our discussion to second-order mappings since we focus on discrete Painlevé equations. We shall use the term first-degree (second-order) mapping for systems of the form

$$A(x_n)x_{n+1}x_{n-1} + B(x_n)x_{n+1} + C(x_n)x_{n-1} + D(x_n) = 0. \tag{2.1}$$

This means that x_{n+1} can be expressed in terms of x_{n-1} through a homographic mapping. A well-known example of first-degree, second-order mapping is the symmetric QRT which has the form

$$f_3(x_n)x_{n+1}x_{n-1} - f_2(x_n)(x_{n-1} + x_{n-1}) + f_1(x_n) = 0. \tag{2.2}$$

The term 'second-degree' mapping is used for systems of the form

$$\begin{aligned} A(x_n)x_{n+1}^2x_{n-1}^2 + B(x_n)x_{n+1}x_{n-1}^2 + C(x_n)x_{n+1}^2x_{n-1} + D(x_n)x_{n+1}^2 + E(x_n)x_{n-1}^2 \\ + F(x_n)x_{n+1}x_{n-1} + G(x_n)x_{n+1} + H(x_n)x_{n-1} + J(x_n) = 0. \end{aligned} \tag{2.3}$$

Evolving this mapping to either forwards or backwards entails taking square roots, thus inducing multivaluedness.

The first example of a discrete Painlevé equation which was presented as a second-degree mapping can be found in a work of the present authors in collaboration with Nijhoff and Papageorgiou [8]. While constructing Lax pairs for discrete Painlevé equations, we obtained a system which we opted to present in the form

$$x_{n+1} + x_{n-1} + 2\frac{z_n - c}{x_n} = \sqrt{(a - x_n - x_{n+1})^2 + 4(z_n + c)} + \sqrt{(a - x_n - x_{n-1})^2 + 4(z_{n-1} + c)} \tag{2.4}$$

where $z_n = \alpha n + \beta$. However, a careful analysis of the results of [8] reveals the fact that (2.4) is nothing but a way to present the equation dubbed asymmetric discrete Painlevé I [9]. The latter has the form

$$x_{n+1} + x_n + y_n = a + \frac{z_n - c}{y_n} \tag{2.5a}$$

$$y_{n-1} + x_n + y_n = a + \frac{z_n + c}{x_n}, \tag{2.5b}$$

where only first-degree mappings appear. However, if we eliminate all reference to y from (2.5) and its up- and down-shifts, we obtain precisely (2.4) for x which is of second degree in x_{n+1} and x_{n-1}. On the other hand, it is quite unnecessary to consider (2.4): if we wish to know the values of x, it suffices to iterate (2.5).

The second example we are going to present also comes from the work of some of the present authors in collaboration with Nijhoff, Satsuma and Kajiwara [10]. Starting from the equation we called the 'alternate discrete Painlevé equation',

$$\frac{z_n}{x_{n+1}x_n + 1} + \frac{z_{n-1}}{x_n x_{n-1} + 1} = -x_n + \frac{1}{x_n} + z_n + \mu, \tag{2.6}$$

we derived its modified form. Putting

$$y_n = \frac{z_n x_{n+1}}{x_n x_{n+1} + 1}, \tag{2.7}$$

we obtained

$$\begin{aligned}(y_{n+1} - y_{n-1})^2 y_n^4 + 2(y_{n+1} + y_{n-1}) y_n^2 (1 - y_n) z_n^2 + z_n^2\big(4y_n^3 + y_n^2\big(z_n^2 - \mu^2 - 4\big)\\ - 2y_n z_n^2 + z_n^2\big) = 0,\end{aligned} \tag{2.8}$$

which is quadratic in $y_{n\pm1}$. The solution proposed in [10] was to show that (2.8) could in fact be obtained by integrating a third-order equation

$$\begin{aligned}y_{n+1}y_n^2 y_{n-1}z_{n-1} - y_{n-2}y_n y_{n-1}^2 z_n + y_n y_{n-1}(z_n - z_{n-1})(y_n y_{n-1} - z_n z_{n-1})\\ - z_n z_{n-1}(y_n z_{n-1} - y_{n-1}z_n) = 0,\end{aligned} \tag{2.9}$$

where y_{n-2} and y_{n+1} enter linearly. A more elegant solution was obtained by Nijhoff [11]. He introduced the auxiliary variables u and v through $u_n = y_n - 1$ and $v_n = x_n u_n + \mu/2$. The latter obey the equations

$$v_n + v_{n+1} = \frac{z_n u_n}{u_n - 1} \tag{2.10a}$$

$$u_n u_{n-1} = v_n^2 - \mu^2/4, \tag{2.10b}$$

and given the form of (2.10*a*), it is straightforward to eliminate the variable u and obtain an equation for v:

$$\left(\frac{v_n + v_{n+1}}{v_n + v_{n+1} - z_n}\right)\left(\frac{v_n + v_{n-1}}{v_n + v_{n-1} - z_{n-1}}\right) = v_n^2 - \mu^2/4, \tag{2.11}$$

which is obviously of first degree in each of the $v_{n\pm1}$.

Two more examples shall be presented in this section, coming from a work of Estévez and Clarkson [12]. They started by examining the Miura of Painlevé II and its relation to the equation known as P_{34} (which can be considered as a 'modified' P_{II}). They obtained the auto-Bäcklund transformations of P_{II} and derived a discrete equation related to, what they called, the potential P_{34}. It has the form

$$(u_{n+1} - u_{n-1})(u_n(u_{n+1} - u_{n-1}) + z_n((u_{n+1} - u_n)(u_n - u_{n-1}) - c)) + z_n^2 = 0 \tag{2.12}$$

where $z_n = n - 1/2$ and $c = t/2$, the quantities n and t being the parameter and the independent variable of the continuous P_{II}, respectively: $x'' = 2x^3 + tx + n$. Equation (2.12) must be considered as a contiguity relation of the solutions of the potential P_{34}. Clearly (2.12) is a second-degree equation and as such should be incompatible with integrability. While this is in principle true if one is talking about *all* the solutions of (2.12), the situation is different if we are interested in the contiguity relations of the potential P_{34}. We start from the relation of u to the solution x of P_{II},

$$x(t,n) = u_n(t) - u_{n+1}(t). \tag{2.13}$$

Using (2.13) in (2.12), we obtain u in terms of x:

$$u_n = \frac{z_n^2}{(x_n + x_{n-1})^2} + z_n \frac{c - x_n x_{n-1}}{x_n + x_{n-1}}. \tag{2.14}$$

Thus, given x_n, one can compute u_n explicitly. In order to obtain x for various values of n, one needs just the contiguity relation of the solutions of P_{II}. The latter was first obtained by Jimbo and Miwa [13] and again by the present authors in collaboration with Fokas [14]. It has the form

$$\frac{z_{n+1}}{x_{n+1} + x_n} + \frac{z_n}{x_n + x_{n-1}} + x_n^2 + c = 0. \tag{2.15}$$

Thus, instead of (2.12), one should consider the system (2.14), (2.15) as the contiguity relation for the potential P_{34}. The advantage of this representation is that one deals only with first-degree mappings.

Estévez and Clarkson also presented the contiguity relation of the solutions of the modified Painlevé IV. Starting from P_{IV} in the form $x'' = x'^2/(2x) + 3x^3/2 + 4tx^2 + 2(t^2 - 2n)x + \mu/x$, they derived its 'modified' analogue and obtained the contiguity relation for its solutions. The latter is a second-degree mapping of the form

$$\begin{aligned}(u_{n+1} - u_{n-1} + 2t)(u_n - u_{n-1})(u_n - u_{n+1})(u_n + 2tz_n) + (u_n - z_n(u_{n+1} - u_{n-1}))^2 \\ - \mu(u_{n+1} - u_{n-1} + 2t)^2 = 0\end{aligned} \tag{2.16}$$

where $z_n = n + 1/2$, the quantities n and t being the parameter and the independent variable of the continuous P_{IV}, respectively. (Note that equation (2.16) as given by Estévez and Clarkson contains some small misprint, which can be traced back to their equation (3.17) and which has been corrected here.) The relation of u to the variable x of P_{IV} is

$$x(t,n,\mu) = u_n(t,\mu) - u_{n-1}(t,\mu). \tag{2.17}$$

However, using (2.17) it is not possible to obtain u in terms of x explicitly, and thus a different approach is necessary. We shall not go into all the details of the derivation which is based on the results we presented in [15]. As shown there, the contiguity relation of the solution of P_{IV} is just the asymmetric discrete Painlevé I, we encountered at the beginning of this section:

$$x_{n+1} = 2t + \frac{z_n - c}{y_n} - x_n - y_n \tag{2.18a}$$

$$y_{n-1} = 2t + \frac{z_n + c}{x_n} - y_n - x_n \tag{2.18b}$$

where c is related to μ through $\mu = c^2$. Using *both* x and y, we can now express u as

$$u_n = (x_n y_n - z_n)(2t - x_n - y_n) + c(y_n - x_n). \tag{2.19}$$

Using (2.18) and (2.19) and eliminating x and y, one can show in a straightforward way that (2.16) is satisfied. On the other hand, (2.19) allows one to bypass the second-degree contiguity (2.16) and obtain directly the solution of the modified P_{IV} for the various values of the parameter n, using only first-degree equations.

3. On q-Painlevé equations obtained from the lattice-modified KdV

In a recent publication [7], Field *et al* considered similarity reductions of the q lattice-modified KdV equation. They started by the multidimensional system

$$a_j v_j v_{ij} + a_i v v_j = a_i v_i v_{ij} + a_j v v_i \tag{3.1}$$

where the index in v indicates that a variable has been up-shifted, i.e. $v_i \equiv v(a_1, \ldots, a_{i-1}, qa_i, a_{i+1}, \ldots, a_M)$. The similarity reduction was obtained using the constraint

$$v(q^{-N}a_1, q^{-N}a_2, \ldots, q^{-N}a_M) = \gamma v(a_1, a_2, \ldots, a_M). \tag{3.2}$$

The authors of [7] considered multi-variable similarity reductions by taking $N = 1$ and coupling several lattice directions. By implementing the constraint on (3.1) for $M = 3$ and $M = 4$, they obtained q-discrete Painlevé equations of second degree. In this section, we shall show that these q-Painlevé equations can be expressed as first-degree systems and moreover correspond to discrete systems already identified.

Before proceeding further, let us proceed to a rapid counting of the degrees of freedom of the reduced systems obtained from (3.1), (3.2). We have M values of a_i, but since (3.1) is homogeneous in a_i, we have in fact $M - 1$ effective parameters. In addition, γ alternates between 'even' and 'odd' values depending on the parity of the number of shifts in v on the rhs of (3.2). For instance, if we define $v(q^{-N}a_1, q^{-N}a_2, \ldots, q^{-N}a_M) = \gamma_e v(a_1, a_2, \ldots, a_M)$, then we have $v(q^{1-N}a_1, q^{-N}a_2, \ldots, q^{-N}a_M) = \gamma_o v(qa_1, a_2, \ldots, a_M)$ and similarly for any odd number of shifts. However, if MN is odd, the two degrees of freedom of γ may also be reduced to a single one by a gauge. Irrespective of the parity of M and N the gauge where we multiply the v's with an even number of shifts by ϕ and those with an odd number of shifts by ψ leaves (3.1) invariant. If MN is even, this gauge has no effect on the γ's since the v's on both sides of (3.2) have a number of shifts of the same parity. On the other hand, if MN is odd, and since we shall focus on the $N = 1$ case, only the parity of M will play a role, the gauge changes γ_e to $\gamma_e\psi/\phi$ and γ_o to $\gamma_o\phi/\psi$. By choosing ϕ and ψ appropriately, we may make the two γ's equal without loss of generality. Thus, when $N = 1$ and M is odd, the total number of variables is just M, while it is equal to $M + 1$ if M is even.

The first interesting case of Field *et al* corresponds to $M = 3$. Here one starts from three copies of the lattice mKdV, involving three parameters, say a, b and c. The reduced system is obtained by considering the evolution along the parameter a, which becomes the independent variable, while b and c become the parameters of the reduced equation. The authors first write the latter as a system of two mappings:

$$\gamma v = w \frac{a\widetilde{\gamma}\widetilde{v} - b\underset{\sim}{w}}{a\underset{\sim}{w} - b\widetilde{\gamma}\widetilde{v}} \tag{3.3a}$$

$$\underset{\sim}{w} = v \frac{aw - c\underset{\sim}{v}}{a\underset{\sim}{v} - cw}, \tag{3.3b}$$

where $w(a, b, c) = v(a, b, q^{-1}c)$ and the tilde in (3.3) indicates an up- or down-shift in a. (According to our explanations in the preceding paragraph, γ and $\widetilde{\gamma}$ play the role of γ_e and γ_o.) Field *et al* proceed from (3.3) to the derivation of a q-Painlevé equation which is of second degree. However, this is due to their particular choice of variables. As we shall now show a different choice, this allows one to obtain a first-degree equation which, moreover, is a well-known one. We start by introducing new variables

$$x = \frac{w}{\gamma v}, \qquad y = \frac{\widetilde{v}}{w}.$$

From (3.3*a*), we find

$$(\gamma\widetilde{\gamma})yx\underset{\sim}{y} = \frac{a+bx}{ax+b}, \tag{3.4a}$$

while from the up-shift of (3.3*b*), we obtain

$$(\gamma\widetilde{\gamma})\widetilde{x}yx = \frac{\widetilde{a}+cy}{\widetilde{a}y+c} \tag{3.4b}$$

where, naturally, $\widetilde{a} = qa$. We remark that as expected from our analysis above, the γ's introduce just one parameter: only the product $\alpha^{-1} \equiv \gamma\widetilde{\gamma}$ appears in the equation. In order to bring (3.4) to a more easily recognizable form, we put $x = u_{2n}$, $y = u_{2n+1}$ and take $z_n = a_0 q^{n/2}$. We can rewrite now (3.4) as

$$u_{m+1}u_{m-1} = \frac{\alpha}{u_m}\frac{z_m + \beta_{e,o}u_m}{z_m u_m + \beta_{e,o}} \tag{3.5}$$

where $\beta_e = b$ and $\beta_o = cq^{-1/2}$, and where the index e or o is determined by the parity of m. Equation (3.5) has three degrees of freedom, as expected for $M = 3$. One recognizes readily the q-$\mathrm{P_{III}}$ equation we derived in [16] in collaboration with Kruskal and Tamizhmani. Equation (3.5) was studied in detail by Kajiwara and Kimura [17] who showed that the geometry of its transformations can be described by the affine Weyl group $A_2^{(1)} \times A_1^{(1)}$.

We turn now to the case $M = 4$. We start from three copies of (3.1) where i is always taken equal to 1 (and $a \equiv a_1$, $b \equiv a_2$, $c \equiv a_3$, $d \equiv a_4$). Moreover for $j = 2$, we write the equation up-shifted in the 3-direction, while for $j = 4$, we down-shifted it in the 4-direction. Down-shifted variables are indicated by an overlined index. We have

$$cv_3v_{13} + avv_3 = av_1v_{13} + cvv_1 \tag{3.6a}$$

$$bv_{23}v_{123} + av_3v_{23} = av_{13}v_{123} + bv_3v_{13} \tag{3.6b}$$

$$dvv_1 + av_{\overline{4}}v = av_{1\overline{4}}v_1 + dv_{\overline{4}}v_{1\overline{4}}. \tag{3.6c}$$

We solve (3.6*a*) for v^3/v^1 to get

$$\frac{v^3}{v^1} = \frac{a+cy}{c+ay} \tag{3.7}$$

where we define $y = v/v_{13}$. Similarly, we define x by v_3/v_{123} and w by $v_{1\overline{4}}/v$, so $\widetilde{w} \equiv w_1 = v_{\overline{4}}/v_1$. Using the similarity relation (3.2), $v_{\overline{1234}} = \gamma_e v$, up-shifted in the 123-directions to $v_{\overline{4}} = \gamma_o v_{123}$, the left-hand side of (3.7) can be expressed as $\gamma_o^{-1}x\widetilde{w}$. So we find

$$x\widetilde{w} = \gamma_o\frac{a+cy}{c+ay}. \tag{3.8a}$$

In the same way, we solve (3.6*b*) for v_{23}/v_{13}, which by the similarity relation is equal to $\gamma_e^{-1}v_{\overline{14}}/v_{13}$ and find

$$wy = \gamma_e\frac{a+bx}{b+ax}. \tag{3.8b}$$

Next, we solve (3.6*c*) for $v/v_{1\overline{4}}$, which can be expressed in terms of $\widetilde{x} \equiv x_1 = v_{13}/v_{1123}$ through the similarity relation as $\gamma_e^{-1}y\widetilde{x}$ and find

$$y\widetilde{x} = \gamma_e\frac{a+d\widetilde{w}}{d+a\widetilde{w}}. \tag{3.8c}$$

A careful treatment of (3.6*c*) down-shifted in the 1-direction leads to

$$\underset{\sim}{y}x = \gamma_o \frac{\underset{\sim}{a} + dw}{d + \underset{\sim}{a}w} \tag{3.8d}$$

where $\underset{\sim}{a} = q^{-1}a$. In the same way, all three equations (3.8*a*)–(3.8c) can be (up and down) shifted in the 1-direction by an arbitrary number of steps, provided the indices of the γ's are interchanged on odd number of shifts, and the a's suitably evolved. Equation (3.8) can be brought to a more easily recognizable form; we put $w = u_{3n-1}$, $x = u_{3n}$, $y = u_{3n+1}$ and take $z_n = a_0 q^{n/3}$. We find

$$u_{m+1}u_{m-1} = \gamma_m \frac{z_m + \alpha_m u_m}{\alpha_m + z_m u_m} \tag{3.9}$$

where γ_m stands for γ_e and γ_o and $\log\alpha = \eta + j^m\kappa + j^{2m}\lambda$, i.e. α reflects the ternary symmetry of b, c, d. Here, since $M(=4)$ is even, we have five degrees of freedom. Equation (3.9), which is obviously of first degree, is a known q-Painlevé equation. It was first obtained in [18], the full complement of its degrees of freedom was presented in [16], while the geometry of its transformations was shown [19] to be described by the affine Weyl group $D_5^{(1)}$.

If one tries to eliminate one of the variables w, x, y in terms of the two others (or, even worse, two variables in terms of a single one), one is led to higher degree mappings like the ones obtained by Field *et al.* However, there exists a possibility of using the homographic character on the rhs of (3.8) to eliminate one variable out of two, for instance whenever a γ_o appears. We thus eliminate w in terms of x and $\underset{\sim}{y}$ and y in terms of x and $\widetilde{w}$. We find

$$\frac{cx\widetilde{w} - a\gamma_o}{ax\widetilde{w} - c\gamma_o}\,\frac{dx\underset{\sim}{y} - \underset{\sim}{a}\gamma_o}{\underset{\sim}{a}x\underset{\sim}{y} - d\gamma_o} = \gamma_e \frac{a + bx}{b + ax} \tag{3.10}$$

and obtained two other equations in a similar way. Note, however, that variables of all three names appear $\ldots, \underset{\sim}{w}, \underset{\sim}{y}, x, \widetilde{w}, \widetilde{y}, \ldots$ though with specific shifts. Equation (3.10), in the appropriate gauge, has been obtained in [20].

4. Conclusion

In this paper, we addressed the question of second-degree discrete Painlevé equations. Our main argument is that, contrary to the situation in the continuous case, second-degree discrete systems (in the sense explained in section 2) are not integrable. This is reinforced by our results in [6] on the non-integrability of correspondences. However, the derivation of the systems examined here would strongly suggest an integrable character. Thus, the present work attempted to lift this apparent contradiction leading to the following interesting result. In every case where a second-degree equation was obtained there existed a transcription to a first-degree system usually through the adequate choice of dependent variables.

There exists one notable exception to our argument on the non-integrability of second-degree mappings. It is related to the QRT mapping [21], which in its symmetric form can be written as

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1}f_2(x_n)}{f_2(x_n) - x_{n-1}f_3(x_n)} \tag{4.1}$$

where the f_i are specific quartic polynomials. The mapping possesses the invariant

$$\begin{aligned}\Phi(x_{n-1}, x_n; K) \equiv{}& (\alpha_0 + K\alpha_1)x_{n-1}^2 x_n^2 + (\beta_0 + K\beta_1)x_{n-1}x_n(x_{n-1} + x_n) \\ &+ (\gamma_0 + K\gamma_1)\left(x_{n-1}^2 + x_n^2\right) + (\epsilon_0 + K\epsilon_1)x_{n-1}x_n + (\zeta_0 + K\zeta_1)(x_{n-1} + x_n) \\ &+ (\mu_0 + K\mu_1) = 0 \end{aligned} \tag{4.2}$$

where K plays the role of the integration constant. As is well known, (4.2) can be parametrized in terms of elliptic functions. The integrability of the QRT mapping is not in contradiction with the fact that correspondences associated with the general biquadratic equation are believed (based on growth arguments) to be non-integrable. If we insist on viewing the QRT mapping as a correspondence, we proceed as follows. Starting from x_{n-1}, x_n, we compute K such that $\Phi(x_{n-1}, x_n; K) = 0$. Having fixed the value of K, the solutions of $\Phi(X, x_n; K) = 0$ are $X = x_{n-1}$ and $X = x_{n+1}$ where the latter is obtained from (4.1). At the next step, solving $\Phi(Y, X; K) = 0$ for both values of X, we do not find four values for Y but only three, namely x_{n+2}, x_{n-2} defined by the up- and down-shift of (4.1) and x_n, the latter being obtained through two different paths which reconnect. Iterating further due to these reconnections the growth of the number of images of a given initial point x is not exponential but linear. Similar arguments can be (and have been [6]) presented for the asymmetric QRT mapping, the integration of which has been given in [22, 23].

Of course, the argument presented here is specific to the QRT mapping and would not apply to other second-degree systems where generically no reconnections take place. The latter considered as correspondences would generically lead to an exponential growth of the number of images (and preimages) of a given point, a feature deemed incompatible with integrability. The approach presented in this paper consisted in salvaging the integrability of the systems examined by rewriting them as first-degree ones in the suitable variables. As a bonus in all the cases examined, the resulting (first-degree) equation was one of the already identified discrete Painlevé equations.

Acknowledgments

AR and BG express their gratitude to P A Clarkson for making reference [12] available to them.

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